The dynamical legacy of Liapunov and Poincaré: Reflections on stability, chaos and randomness

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A personal trajectory

From nonlinear vibes, solid and fluid mechanics and nonlinear optics to animal locomotion, neuroscience, and cognitive psychology.

Dedicated to former students, postdocs and colleagues in Cornell’s Department of Theoretical and Applied Mechanics.
Contents

I: Some history

II: What is stability?
What should we mean by stable? Liapunov’s methods, counterexamples, exponents, and some examples.

III: More exotic and random stabilities
Noisy heteroclinic cycles, regularity in chaos & stochastic dynamics: tracking turbulence and modelling the brain.

A taste of history, some theory, & applications from various fields.
I: A little history

(Inspired by fluid mechanics and astronomy, two mathematicians start working. They set the scene, but they leave plenty for the rest of us do ... )
Liapunov’s advisor, P. Chebyshev, proposed that he study equilibria of rotating fluid bodies. This he did in his MSc thesis (St. Petersburg, 1884), proving stability by showing that the equilibria are minima of a “modified” potential energy. This led to the general stability results of his PhD thesis (Moscow U, 1892), written while teaching at Kharkov U.

In the thesis, Liapunov proposed his first and second methods, characteristic numbers (now called Liapunov exponents), and, almost, center manifolds (the L center theorem).

Liapunov was aware of Poincaré’s work ‘on the curves defined by differential equations’ (1881-86), and credited it as guiding his own investigations.

Later, L worked in probability theory (CLT), suggested by his teaching duties (he was a friend of Markov at school and St. Petersburg U.).
Liapunov’s thesis, 1892
The general problem of the stability of motion.

242 pp:
the first textbook
in Stability Theory.

tr into French 1907,
tr into English 1992!

Kharkov Mathematical Society, 1892
(Int. J. Control 55 (3), 531-573, 1992)
King Oscar’s Prize, 1885-1890

KING OSCAR’S PRIZE
(Problem 1 of 4)

Given a system of arbitrarily many mass points that attract each other according to Newton’s laws, try to find, under the assumption that no two points ever collide, a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

This problem, whose solution would considerably extend our understanding of the solar system, would seem capable of solution using analytic methods presently at our disposal; we can at least suppose as much, since Lejeune Dirichlet communicated shortly before his death to a geometer of his acquaintance [Leopold Kronecker], that he had discovered a method for integrating the differential equations of Mechanics, and that by applying this method, he had succeeded in demonstrating the stability of our planetary system in an absolutely rigorous manner. Unfortunately, we know nothing about this method, except that the theory of small oscillations would appear to have served as his point of departure for this discovery. We can nevertheless suppose, almost with certainty, that this method was based not on long and complicated calculations, but on the development of a fundamental and simple idea that one could reasonably hope to recover through preserving and penetrating research.

In the event that this problem nevertheless remains unsolved at the close of the contest, the prize may also be awarded for a work in which some other problem of Mechanics is treated in the manner indicated and solved completely.

Poincaré’s prize memoir, 1889-90:
On the three-body problem and the equations of dynamics.

Acta Mathematica 13, 1-270, 1890. It led to:
Les méthodes nouvelles de la mécanique céleste (3 vols), 1892, 93, 99; 1272 pp.

270 pp: the first textbook in *Dynamical Systems*.
(never tr)

MNMC tr into English 1993!
## Planting the seeds for a century’s nonlinear dynamics

<table>
<thead>
<tr>
<th>Liapunov</th>
<th>Poincaré</th>
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<td>stability</td>
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Characterized as:  “... organized ...”

“... a patchwork of ... tools and results ... but its scope is considerably wider ...”

From the committee’s report on Poincaré’s PhD thesis:

.... the remainder of the thesis is a little confused and shows that the author was still unable to express his ideas in a clear and simple manner. Nevertheless, considering the great difficulty of the subject and the talent demonstrated, the faculty recommends that M Poincaré be granted the degree of Doctor with all privileges.
Liapunov and Poincaré were both motivated by Celestial Mechanics (Liapunov’s father was an astronomer).

Newton and Euler (17–18th c) integrated the differential equations for two bodies (Sun + Earth) and found the elliptical orbits of Kepler; they showed that the inverse square law also predicted Kepler’s first and third laws. They found celestial order:

Conservation of linear and angular momenta and energy

\[ \mu r^2 \dot{\theta} \overset{def}{=} L_0 = \text{const.} \]

\[ \frac{\mu}{2} r^2 + \frac{L_0^2}{2 \mu r^2} + V(r) = E = \text{const.} \]

Real space: Kepler’s conic sections

Phase space
The three body problem is much harder

Later, Newton struggled unsuccessfully with the “problem of the moon” (sun, earth, and moon). This was idealised as the restricted, planar, circular, 3-body problem:

\[ \theta = \omega t \]

Newton was unable to solve it, nor could Euler, Lagrange, Laplace, Poisson, Liouville, &c. &c., and Poincaré was not entirely successful either ... In fact he discovered chaos (homoclinic tangles), thereby showing that it was, in the classical sense, insoluble.

Dynamics problems are difficult!
Surely, the (simple) pendulum is simple?

If we keep the pivot still, we get Equilibria and stable periodic orbits:

With a motionless support, as in the two-body problem, conservation of energy

\[ E = \frac{ml^2 \omega^2}{2} + mg l (1 - \cos \theta) = \text{const.} \]

implies nested sets of periodic orbits and a separatrix (a homoclinic orbit).
But with a small periodic force, it’s not so simple!

The separatrix splits so that orbits can wander from oscillations to rotations and back, giving sensitive dependence on initial conditions and chaos!
II: What is stability?
(or, the importance of being careful)
A zoo of solutions! How to start?

Dynamics is hard. So ask simpler questions. Classify special kinds of solutions: equilibria (fixed points), periodic orbits, other, more complicated non-wandering and recurrent sets: tori, fractals, strange attractors ...

They’re easier to find than the entire global phase portrait (the set of all possible solutions), but are they relevant? Do they describe observable behaviors? Are they stable to (sufficiently small) perturbations?

And what does it mean to be stable anyway?
Liapunov stability of fixed points

One must be careful*: it’s necessary to define two regions $U$ and $V$, with $V$ inside $U$, such that if you start in $V$ you can’t escape from $U$, and such that $U$ and $V$ can be taken as small as you wish, while still containing the equilibrium point $x^0$.

Definition 1 (Liapunov stability) $x^0$ is a stable fixed point of (4.1) if for every neighborhood $U \ni x^0$ there is a neighborhood $V \subseteq U$ such that every solution $x(t)$ of (4.1) starting in $V$ ($x(0) \in V$) remains in $U$ for all $t \geq 0$. If $x^0$ is not stable, it is unstable.

Definition 2 (Asymptotic stability) $x^0$ is asymptotically stable if it is stable and additionally $V$ can be chosen such that $|x(t) - x^0| \to 0$ as $t \to \infty$ for all $x_0 \in V$.

* Explosions, non-normal matrices and transient growth.

So, how can we find such regions?
Linearization (L’s first method)

Since we can solve all linear constant-coefficient differential equations, why not just linearize at the equilibrium in question and ask if solutions of the linearized system are stable? This should be OK for “small, nearby solutions,” right? Many believed so (Laplace, Lagrange, …), but even for conservative systems, it’s tricky. Nonlinear resonances can cause solutions to escape!

Soluble example: A 2:1 Hamiltonian resonance with cubic terms (T. Cherry, H. Pollard)

\[
H = \frac{(q_1^2 + p_1^2)}{2} - (q_2^2 + p_2^2) + \frac{(p_1^2 p_2 - p_2 q_1^2 - 2q_1 q_2 p_1)}{2}
\]

\[
q_1 = -\frac{\sqrt{2} \cos(t - t_\ast)}{(t - t_\ast)}, \quad q_2 = \frac{\cos 2(t - t_\ast)}{(t - t_\ast)}, \quad p_1 = \frac{\sqrt{2} \sin(t - t_\ast)}{(t - t_\ast)}, \quad p_2 = \frac{\sin 2(t - t_\ast)}{(t - t_\ast)}
\]

Picking \(t_\ast\) large, starting as near to equilibrium \((0,0,0,0)\) as you wish, the solution blows up as \(t \to t_\ast\)! This Hamiltonian is not positive-definite.

Building on Poincaré’s studies of 2- and 3-dimensional systems in the 1880’s, Liapunov gave the general conditions sufficient for linearization to work in both conservative and more general systems.
The importance of (counter-)examples

As we’ve seen, Liapunov credited Poincaré’s 1880’s work, but L and P corresponded only on the figures of equilibrium for rotating fluids (L’s master’s thesis project), not on stability per se. However, Liapunov did write to P’s colleague Emile Picard in 1895, regarding his ‘Traité d’analyse’ (also 3 vols!), providing examples such as:

\[
\begin{align*}
\dot{x} &= -y + x(x^2 + y^2) \\
\dot{y} &= x + y(x^2 + y^2)
\end{align*}
\]

This showed that Picard’s claims of existence of families of periodic orbits based on the linearization were false, without the additional condition of a first integral (e.g., a Hamiltonian). It’s also the Poincaré-Hopf normal form.

Local asymptotic stability by linearization

\[ \dot{x} = f(x), \quad x^0 \text{ fixed } (f(x^0) = 0) \Rightarrow \dot{\xi} = Df(x^0)\xi \]

If all the eigenvalues of the Jacobian matrix \( Df(x^0) \), defining the linearized system at \( x^0 \) have **strictly negative real parts**, then it is **locally asymptotically stable**. Such fixed points are called **hyperbolic** (an unfortunate name, derived from the shape of solutions near unstable, non-degenerate saddle points).

This extends to discrete systems and Poincaré maps, and thus to local asymptotic stability of hyperbolic periodic orbits (Floquet theory and characteristic multipliers, cf. Poincaré). More general hyperbolic attractors, in which orbits remain but locally separate exponentially fast, can be described by **Liapunov exponents**, the long time limits of growth/decay along the linearized flow:

\[ \lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\Delta \xi(t)}{\Delta \xi(0)} \right| \quad \text{(max. L. exp.)} \]
Stability of sets of orbits does not imply L stability

**Definition 3** Two orbits $\gamma(x)$ and $\gamma(\dot{x})$ are $\epsilon$-close if there is a reparameterization of time (a smooth, monotonic function) $\tilde{t}(t)$ such that

$$|\gamma_t(x) - \gamma_{\tilde{t}(t)}(\dot{x})| < \epsilon \text{ for all } t \geq 0.$$ 

We then say that an orbit is orbitally stable if all orbits with nearby initial points remain close in the sense of Definition 3:

**Definition 4** An orbit $\gamma(x)$ is orbitally stable if, for any $\epsilon > 0$, there is a neighborhood $V$ of $x$ so that, for all $\dot{x} \in V$, $\gamma(x)$ and $\gamma(\dot{x})$ are $\epsilon$-close.

**Definition 5** If additionally $V$ may be chosen so that, for all $\dot{x} \in V$, there exists a constant $\tau(\dot{x})$ so that

$$|\gamma_t(x) - \gamma_{t - \tau(\dot{x})}(\dot{x})| \to 0 \text{ as } t \to \infty.$$ 

then $\gamma_t(x)$ is asymptotically orbitally stable.

An example due to J. Milnor. The 2-tori are close, not so the individual orbits: they drift in and out of phase as time rolls by.

$$\begin{align*}
\dot{\theta}_1 &= 0 \\
\dot{\theta}_2 &= \sin \theta_1 \\
\dot{\theta}_3 &= 1
\end{align*}$$

(Also, Milnor’s definitions of attractors.)

Liapunov functions (L’s second method)

Liapunov’s direct or second method: avoid resonance and h.o.t. problems! To prove that solutions stay within a bounded region, don’t try to solve the equation, just build a wall around the fixed point and show that no one can cross it. Better yet, build a series of nested walls and show that everyone must keep crossing them inward. Many dynamical systems are a mix of these two important special cases:

1. **Conservative systems**  
   (Newtonian mechanics, \( F = ma \))

   \[ \frac{\mu}{2} r^2 + \frac{L_0^2}{2\mu r^2} + V(r) = E = \text{const.} \]

   Solutions run around on the walls  
   **Liapunov function** = total mechanical energy.

2. **Gradient systems**  
   (Aristotelian mechanics, \( F = cv \))

   \[ \dot{x} = -\nabla V(x) \]

   **Liapunov function** = potential \( V(x) \).  
   Solutions cross the walls at right angles
Casimir functions: using conserved quantities

Sometimes, due to symmetries of the physical system (e.g., translation, rotation), quantities such as linear and angular momentum are conserved as well as the total energy. Liapunov functions can be built by forming nonlinear functions of these conserved quantities and adding them to create a grand positive-definite nest around the equilibrium. (Especially handy when Hamiltonian energy is not pos-def!)

Example 1: Kirchhoff eqns for bodies under (ideal, inviscid, irrotational) water: added fluid mass $M$ and inertia $I$ couples translation and rotation dynamics.

$$
\dot{p} = p \times \omega, \quad \dot{\pi} = \pi \times \omega + p \times v, \\
p = Mu, \quad \pi = I\omega; \\
H = \frac{[p \cdot M^{-1}p + \pi \cdot I^{-1}\pi]}{2}, \\
C_1 = p \cdot \pi, \quad C_2 = |p|^2; \\
V(p, \pi) = H + \Phi(C_1, C_2). \\
\gamma = C_1/C_2 = \frac{\text{linear mom.}}{\text{angular mom.}}.
$$

Good choices of $V$ reveal stable branches; body spinning about different axes.

Proof = Liapunov!

Example 2: computation in neural networks

In content-addressable memories, like our own, one wants to recall a specific memory (e.g., of a face) stably, based on a partial or noisy cue. Can a network of N leaky, mutually-inhibiting ‘neural’ units do this? Yes, if the coupling matrix $T_{ij}$ is symmetric.

**Proof = Liapunov!**

The ‘Hopfield energy’ $E$ provides a Liapunov-like function, in 2-d it’s a double-well potential.

$$
\dot{u}_i = -u_i/R_i + \sum_j T_{ij} g_j(u_j) + I_i
$$

$$
E = -\frac{1}{2} \sum_{i,j} T_{i,j} g(u_i) g(u_j) - \sum_i \frac{1}{R_i} \int_{g^{-1}(0)} g(u) u_i g(u_i) du + \sum_i I_i g(u_i)
$$

Synaptic coupling is typically sigmoidal:

III: More exotic and random stabilities

(invariant measures, heteroclinic cycles and stochastic dynamical systems, or order in chaos, coherent structures in turbulence, and the stable brain)
Poincaré’s deterministic chaos revisited

Duffing’s nonlinear oscillator seems to have a strange attractor supported on a fractal set:

\[ \ddot{x} + \delta \dot{x} + x - x^3 = A \cos(\omega t), \quad \delta = 0.2, \ A = 0.3, \ \omega = 1. \]

chaotic and yet ordered.

Order in chaos: invariant probability measures

If we can’t predict exactly what will happen, it’s still useful to predict what’s likely to happen. A simple example: the logistic equation (a population of bugs in a box): $x_{n+1} = \lambda x_n (1 - x_n)$. For $\lambda$ ‘near 4,’ chaotic attractors exist, but they display much statistical regularity.

Micro ‘logistic attractors’ like these appear near transverse homoclinic points, partially explaining Duffing’s attractor.

D. Ruelle, J. Palis, M. Viana, L-S. Young, ... 1980-2000’s.
The surprising stability of heteroclinic cycles

Poincaré discovered heteroclinic orbits: solutions connecting unstable saddle points. They are a central element in chaos: remember the pendulum! In systems with translation or rotation symmetries, a chain of heteroclinic orbits can form a cycle that is robust against perturbations: it is structurally stable! If the saddles are dissipative*, these cycles can be asymptotically stable.

Saddle-sink connections on invariant planes are stitched together to form a closed cycle. An integrable limit reveals that a branch of periodic orbits (modulated traveling waves) bifurcates from the cycle, changing its stability.

*Negative e-values beat positive e-values.

Coherent structures in turbulence

In deterministic chaotic systems many things can happen, but not everything. Turbulent fluids are similar: many flows are dominated by coherent structures: spatio-temporal features that recur, more or less regularly, often visiting unstable equilibria or periodic orbits. How come, and why come again and again? They’re built on asymptotically and structurally stable symmetric heteroclinic cycles!

Plane Couette flow in physical space and projected into phase space

Coherent vortices in a turbulent flow collapse and reform as solutions of the Navier-Stokes equations visit and revisit unstable states.
Noisy heteroclinic attractors

Solutions approaching stable heteroclinic cycles spend longer and longer hesitating near saddle points, as if a periodic orbit were slowing down. Additive random noise produces characteristic time scales as orbits are nudged past the saddles. Such attractors can be proved to exist in low-dimensional models of turbulence.

\[ \langle T \rangle = k_1 + \frac{(|\log \sigma| + k_2)}{\lambda_u} \]
\[ \sigma = \text{s.d. of white noise.} \]

Five (complex) mode model of a turbulent boundary layer. Noise enters from the outer flow, maintaining the burst-sweep cycle in which metastable vortices form and collapse, form and collapse ...

Distribution of interburst intervals

And now, for something completely different ...

or maybe not so different.
Noisy brains: a simple perceptual decision task

“On each trial you will be shown one of two stimuli, drawn at random. To earn a reward, you must correctly identify the direction (L or R) in which the majority of dots are moving.” The experimenter can vary the coherence of movement (% moving L or R) and the delay between response and next stimulus. “Your goal is to maximize net rewards over many trials in a fixed period.”

30% coherence

5% coherence

Courtesy: W.T. Newsome, Stanford Univ.

Behavioral measures: reaction time distributions, accuracy and speed-accuracy tradeoff.
A model for decision making with noisy evidence

Biophysical spiking neuron models reduce to a pair of leaky competing accumulators:

\[
\begin{align*}
\dot{y}_1 &= [-\gamma y_1 + f(-\beta y_2) + s_1] dt + \sqrt{D} dW_1 \\
\dot{y}_2 &= [-\gamma y_2 + f(-\beta y_1) + s_2] dt + \sqrt{D} dW_2 
\end{align*}
\]


The difference of the accumulating evidences describes a 1-d drift-diffusion process with decision variable \( x = y_1 - y_2 \):

\[
\dot{x} = \Delta I dt + \sigma dW
\]

Stochastic center manifold

L. Arnold et al. 1990-2000's;
Dynamics of the drift-diffusion (DD) process

Reaction time data can be fitted to the first passage threshold crossing times of a DD process: \(dx = \Delta I \, dt + \sigma \, dW\). It also describes neural activity.

Frontal eye field (FEF) recordings in monkeys.

J. Schall et al., Neuron, 2002;  
Noisy systems can be (stochastically) stable

If leak is bigger than inhibition, the one-dimensional system, evolving on its stochastic center manifold, is a stable Ornstein-Uhlenbeck process. Sample paths clump together to form an evolving (Gaussian) distribution $p(x, t)$.

\[
\begin{align*}
\dot{x} &= \Delta I - \lambda x \Rightarrow x(t) \rightarrow \Delta I / \lambda \\
\frac{dx}{dt} &= (\Delta I - \lambda x) \, dt + \sigma \, dW
\end{align*}
\]

Fokker-Planck-Kolmogorov equation: predicts evolving PDF $p(x, t)$

Cued response task: another way of viewing the evolving probability of choosing 1 or 2:

The distribution stabilizes with mean $\frac{\Delta I}{\lambda}$ and variance $\frac{\sigma^2}{2\lambda}$.
The DD model predicts psychometric functions

The DD/OU model predicts accuracy as a function of stimulus viewing time $T$ and coherence $C$:

$$dx = (\Delta I(C) - \lambda x) dt + \sigma dW$$

$$P_{\text{corr}}(C, T) = \int_0^T p(x, T; C) dx$$

$$= \frac{1 + \text{erf}[b_1(C + b_2)]}{2}$$

Comparison with primate data:
Probability of detecting stimulus 1 or 2: the model fits the data beautifully!

It’s not just a good description, it also explains mechanisms in the brain via direct neural recordings, reduction of spiking models to low-dimensional systems, bifurcation and stability analyses, ...

Different field, similar stabilities!

The morals of the story

• Theoretical and mathematical advances are often driven by practical problems and examples (from celestial, conservative, and dissipative mechanics to biology and neuroscience).
• Details matter; the ‘right’ definitions and concepts depend upon the application; theory feeds on practice.
• Abstract examples and counterexamples are important.
• What you can do depends on what you know; what you will learn depends on what you try to do.
• Deterministic and stochastic systems are not so far apart.
• One person’s chaos is another’s stability …
• There’s still plenty to do (e.g., piecewise-smooth systems, freeplay, …)

~ The End ~
Thanks for your attention, and thanks to Liapunov and Poincaré for starting us off so well!